CONDITIONS FOR THE EXISTENCE OF A PERIODIC SOLUTION OF A THIRD-ORDER DIFFERENTIAL EQUATION

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Let us consider the third-order differential equation

$$\dddot{x} + \alpha \ddot{x} + \beta \dot{x} + \sin x = e(t) \tag{1}$$

Here α and β are positive constants, $e(t)$ is a square-integrable periodic function of period 2π. This equation is encountered, in particular, in the investigation of synchronous elements in television [1].

In this note the author derives, on the basis of a theorem proved by Barbashin [2], a criterion for the existence of a periodic (in t) solution of Equation (1).

Equation (1) is equivalent to the system of differential equations

$$\dot{x} = y, \qquad \dot{y} = z, \qquad \dot{z} = -\alpha z - \beta y - \sin x + e(t) \tag{2}$$

We introduce the notation

$$\varphi(x) = x - \sin x$$

Then the system of first approximation takes the form

$$\dot{x} = y, \qquad \dot{y} = z, \qquad \dot{z} = -x - \beta y - \alpha z \tag{3}$$

Let us assume that the origin is a stable singular point of the system (3) of the type of a "generalized focus", i.e. we assume that the characteristic equation

$$r^3 + \alpha r^2 + \beta r + 1 = 0$$

of the system (3) has one real root $\lambda < 0$, and two complex roots $a \pm bi$, where a and b are real numbers, $a < 0$, $b > 0$. From the hypothesis that the origin is a stable point for the system (3), it follows that $\alpha\beta > 1$.

Let $\mu = \max(a, \lambda)$, i.e. μ is the larger one of the numbers a and λ. We consider the number

$$B^\circ = \frac{M + (a^2 + b^2 + 1) N}{\Delta}$$

where

$$\Delta = b\,[(a - \lambda)^2 + b^2], \quad M = b\,[(a - 1)^2 + b^2] \quad N = \sqrt{\frac{\Delta}{b}}\,(1 + \sqrt{-\lambda} + \sqrt{\lambda^2 + \beta}$$

Suppose that the following conditions are satisfied in a region D of the x, y, z-space:

$$t \geqslant 0, \qquad \max(|x|, |y|, |z|) \leqslant \varepsilon, \qquad \varepsilon = \sqrt{\frac{-2\mu}{B^\circ}}$$

Barbashin's [2] theorem, which establishes the existence of a periodic solution, is applicable to a system of differential equations if a number of conditions are satisfied by that system. For the system (2), in addition to the conditions already indicated, one such restriction is the existence of a fundamental matrix $W(t, \tau) = \| w_{ik}(t, \tau) \|$ $(i, k = 1, 2, 3)$ for the system (3) satisfying the conditions

$$W(\tau, \tau) = E, \qquad \| W(t, \tau) \leqslant Be^{\mu (t-\tau]} \tag{4}$$

Here E is a unit matrix, B is a positive constant, $\mu < 0$.

The evaluation of the fundamental matrix of the solution of system (3) which becomes the unit matrix when $t = \tau$, does not present any difficulties. Its elements are, obviously, given by

$$w_{1k} = \frac{1}{\Delta}\{\Delta_{1k} e^{\lambda (t-\tau)} + e^{a (t-\tau)} [\Delta_{2k} \cos b(t - \tau) + \Delta_{3k} \sin b(t - \tau)]\}$$

$$w_{2k} = \frac{dw_{1k}}{dt}, \qquad w_{3k} = \frac{d^2 w_{1k}}{dt^2}$$

where

$$\Delta_{11} = -\frac{b}{\lambda}, \qquad \Delta_{21} = b\lambda(\lambda - 2a), \qquad \Delta_{31} = \lambda(a^2 - b^2) - \lambda^2 a$$

$$\Delta_{12} = -2ab, \qquad \Delta_{22} = 2ab, \qquad \Delta_{32} = -a^2 + b^2 + \lambda^2$$

$$\Delta_{13} = b, \qquad \Delta_{23} = -b, \qquad \Delta_{33} = a - \lambda$$

In order to obtain an estimate of the norm $\| W(t, \tau) \|$ of a matrix, we define the norm of an arbitrary vector $X = (x, y, z)$ and that of a matrix as

$$\| X \| = \max \quad (| x |, | y |, | z |) \qquad \| W (t, \tau) \| = \max_{1 \leqslant i \leqslant 3} \sum_{k=1}^{3} | w_{ik} |$$

It is easy to show that in the region D the following inequalities are satisfied:

$$\begin{aligned} &\sum_{k=1}^{3} | w_{1k}(t, \tau) e^{-\mu (t-\tau)} | \leqslant \frac{M + N}{\Delta} \\ &\sum_{k=1}^{3} | w_{2k} (t, \tau) e^{-\mu (t-\tau)} | \leqslant \frac{M + \sqrt{a^2 + b^2} N}{\Delta} \\ &\sum_{k=1}^{3} | w_{3k} (t, \tau) e^{-\mu (t-\tau)} | \leqslant \frac{M + (a^2 + b^2) N}{\Delta} \end{aligned} \tag{5}$$

For the constant B, which estimates the norm $\| W(t, \tau) e^{-\mu(t-\tau)} \|$ and which occurs in the condition (4), one should select the number determined by the equation

$$B = \frac{M + N}{\Delta} \quad \text{when } a^2 + b^2 \leqslant 1, \qquad \frac{M + (a^2 + b^2) N}{\Delta} \quad \text{when } a^2 + b^2 > 1$$

The next restriction on the system (2), which is required for the validity of the application of Barbashin's theorem, is the existence of a Lipschitz constant for the function $\phi(x)$ which satisfies the relation $(-\mu - LB) > 0$.

In the considered region D, the Lipschitz constant for the function $\phi(x) = x - \sin x$ is the number

$$L = \frac{\varepsilon^2}{2} = \frac{-\mu}{B^\circ}$$

It is easily seen that it satisfies Equation (5). From here on we shall denote the left-hand side of Equation (5) by r. Then

$$\gamma = -\mu \frac{B^\circ - B}{B^\circ} > 0$$

From what has been said we may conclude that the conditions of the theorem of Barbashin are satisfied [1]. Thus, we may state the following result for Equation (1).

Theorem. Suppose that conditions (4), (5) and

$$(A) \qquad \sup_{0 \leqslant t \leqslant 2\pi} | e (t) | < \frac{\varepsilon \gamma}{2B^2}$$

$$(B)\qquad \int_0^{2\pi} |e(t)|\,dt < \frac{\varepsilon}{2B^2} e^{-2\pi\gamma}(1 - e^{-2\pi\gamma})$$

$$(C)\qquad \left(\int_0^{2\pi} e^2(t)\,dt\right)^{\frac{1}{2}} < \frac{\varepsilon}{2B^2} \left(\frac{2\gamma}{e^{4\pi\gamma} - 1}\right)^{\frac{1}{2}} (1 - e^{-2\pi\gamma})$$

are satisfied for the system (2).

Let $\delta = \epsilon/2B$. Then the following statements are true.

1) Every solution $X(t)$ of the system (2) is such that $\|X(t_0)\| \leqslant \delta$ does not leave the region D if $t \geqslant t_0$.

2) There exists a number $T > t_0$ such that $\|X(t)\| \leqslant \delta$ if $\|X(t_0)\| \leqslant \delta$ and $t > T$.

3) In the region D there exists an asymptotically stable periodic trajectory which attracts all other trajectories issuing from the region $\|X\| < \delta$ when $t = t_0$.

The numbers ϵ, B and γ are given above.

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